Additive Growth*

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March 2022

Abstract

Growth theory is based on the assumption of exponential total factor productivity (TFP) growth. Across countries and time periods I find that TFP growth better described as an additive stochastic process. The additive growth model offers natural explanations for the TFP slowdown and volatility puzzles, and for falling interest rates. For the distant future the model predicts ever increasing increments to standards of living but with growth rates that converge to zero. For the distant past the model suggests that the size of TFP increments changes around the first and second industrial revolutions.

*I am grateful to Chad Jones, Greg Mankiw, Xavier Gabaix, Virgiliu Midrigan, Xavier Jaravel, Antonin Bergeaud, Remy Lecat, and Callum Jones.

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Following Solow (1956) the textbook model assumes that TFP growth follows a geometric process where the size of the next increment is proportional to the level of TFP. I examine data across many countries and time periods and I find that, in nearly all cases, productivity growth is in fact arithmetic. For instance, using data from Bergeaud et al. (2016), US TFP growth after World War 2 is well described by the following statement: Hicks-neutral TFP, normalized to 1 in 1947, increases by about 2.45 percentage points each year for more than 80 years. Using data from Fernald (2012) for the *private* sector, the same statement holds with an annual increments of 2.76 percentage points.

I start my investigation with the US using data from Fernald (2012) – "Fernald" – and Bergeaud et al. (2016)– "BCL". I find that TFP growth is linear during the post war period. The inverse marginal product of capital is also linear, as predicted by the first order condition for capital demand. And the linear TFP model predicts the correct non-linear evolution of labor productivity. For long-term US TFP growth (1890-2019) the additive growth model's 10-year forecast errors are 25% to 45% lower than those of the exponential growth model.

I then consider the 23 countries in the BCL sample. I find that TFP dynamics are better described by the additive model for all of the 23 countries in the post war sample and in the long sample (1890-2019). I also consider a sample of OECD countries (e.g., Korea) that are not in the BCL sample and I show that their TFP growth has also been linear.

The exponential growth model fails in two related dimensions. The first is that it predicts periods of exponential productivity growth that simply do not exist in the data. The second is that the trend growth rates are unstable. Tests of structural breaks in the exponential model find a large number of breaks. By contrast, the additive TFP model displays few breaks and, at least in some cases, these breaks have a plausible economic interpretation in terms of General Purpose Technologies (GPTs). For example, the process of US TFP increments has only one break over the past 130 years, immediately after the Electrification revolution (Gordon, 2016; Jovanovic and Rousseau, 2005).

The theoretical properties of the additive TFP model are simple. The model has a balanced growth path with a constant capital/output ratio. The capital labor ratio, labor productivity, and GDP per capita grow indefinitely, and with increasing increments. The model therefore does not predict stagnation: incomes are increasing ever faster even as growth rates tend to zero.

Literature The literature on growth is enormous and cannot be summarized here. My result shed new light on existing puzzles in the following areas. The first area is growth accounting, from Solow (1956) to Hall and Jones (1999) and Gordon (2016). I make two contributions here: I argue that the perceived TFP slowdown is the result of a misspecified model, since growth was never actually geometric, and I show that a well specified model provides useful benchmarks and forecasts across a wide variety of countries and time periods.

The second area concerns the sources of growth. The additive model sheds new light on the role of human capital accumulation, as in Mankiw et al. (1992), and on the distinction between Hicks-neutral and Harrod-neutral progress.

Finally my results speak to the correct specification of models of endogenous growth, such as Romer (1986b), Lucas (1988) and Aghion and Howitt (1992). These models assume a knowledge production function that delivers a constant growth rate. My results suggest that the knowledge production function should deliver constant productivity increments instead, with changes in the size of the increments happening only around the discovery of new GPTs. These insights relate to Jones (2009) and to the recent work of Bloom et al. (2020) on the declining productivity of research and development activities.[1] If the correct growth model displays constant increments instead of a constant growth rate, the research productivity puzzle is two or three times smaller than previously thought.

1 Evidence from US Growth

My main sources of data – Fernald (2012) and Bergeaud et al. (2016) – assume a Cobb-Douglas production function, so I will do the same in most of the paper. Aggregate value value added (GDP, Y_t) is given by

$$Y_t = A_t K_t^{\alpha} L_t^{1-\alpha}, \tag{1}$$

where A_t is TFP, K_t is the flow of capital services and L_t is the flow of labor services. Section 3 discusses more general production functions $F(K, L, A)$ and compares Hicks and Harrod neutrality in the context of linear growth. My goal is to understand the

[1] Alexey Guzey, in a blog post, has criticized Bloom et al. (2020)' assumption of exponential growth as a benchmark for measuring labor productivity. See https://guzey.com/economics/bloom/#bloom-et-al-appear-to-not-realize-that-most-of-the-data-they-analyze-in-the-paper-including-the-us-tfp-does-not-exhibit-exponential-growth.

long-term dynamics of TFP. Since at least Solow (1956) economists have assumed that A follows a geometric process, which I call model G[2]:

$$\mathbb{E}\left[A_{t+\tau} \mid A_t\right] = A_t \left(1+g\right)^{\tau}. \tag{2}$$

I am instead arguing that growth is additive and that the TFP process is better described by model D (as in "difference"):

$$\mathbb{E}\left[A_{t+\tau} \mid A_t\right] = A_t + \tau\Delta, \tag{3}$$

where Δ is a parameter that measures the size of increments. I start my investigation with post-war US data for two main reasons, one empirical, one theoretical. The empirical reason is the this is the most reliable and most widely used data. The theoretical reason is that one would expect different TFP dynamics between countries at the frontier and countries catching up to the frontier. The main advantage of post-war US data, then, is that one can reasonably argue that the US was at the technological frontier during the entire period.

1.1 Main Data Sources

My primary sources for TFP are Fernald (2012) (Fernald) and Bergeaud et al. (2016) (BCL). Let A_t^{BCL} and A_t^F denote the BCL and Fernald measures of TFP. There are several differences between these two datasets. BCL covers 23 countries from 1890 to 2019 and their data allow the analysis of a long sample as well as international comparisons in Section 2. Fernald's series cover only the US business sector, while BCL include households and the government. Fernald includes an adjustment for capacity utilization to make the series comparable to the theoretical benchmark. Finally, Fernald also includes an adjustment for human capital, following Mankiw et al. (1992). Formally, BCL assume that $L_t = H_t$, total hours worked, while Fernald assume $L_t = Q_t H_t$ where Q_t is an index of labor quality based on education. Using (1), we see that the Fernald's measure comparable to the BLC measure is

$$A_t^{F,NQ} = A_t^F Q_t^{1-\alpha}$$

[2]Jensen's inequality terms do not play a significant role in the empirical analysis of this section.

where A_t^F is Fernald's labor-quality-adjusted TFP measure. Figure 8 in the Appendix shows the three TFP series, where A_t^{BCL} is normalized to 1 in 1947 to be comparable with Fernald's measures. The key point is that none of the series is well described by the exponential process (2) with constant g. $A_t^{F,NQ}$ and A_t^{BCL} are well described by the additive process (3) with constant Δ. Fernald's unadjusted measure $A_t^{F,NQ}$ grows somewhat faster than A_t^{BCL} because productivity growth is faster in the business sector than in the rest of the economy. For the A_t^F there is some slow down in the later part of the sample even according to (3) because some of the measured productivity gains are attributed to the labor quality factor. This speaks to the model specification issue that I discuss in details in Section 3.

1.2 Postwar U.S. TFP

The simplest way to start comparing model D and model G is to consider the following experiment. Suppose that two agents, *George* and *Daniela*, are asked in the middle of the sample (1983) to predict the level of TFP in the second half of the sample (1984-2019). The agents have access to data from the end of World War 2 until 1983. The two agents have dogmatic beliefs regarding the correct model of economic growth. George believes in model G from equation (2) while Daniela believes in model D from equation (3). George therefore fits a log linear model $\log(A_t) = \hat{\alpha}_g + \hat{g}t$ over the years 1947 : 1983 and predicts future (log) TFP as

$$\log\left(\hat{A}_t^{(G)}\right) = \hat{\alpha} + \hat{g}_A t$$

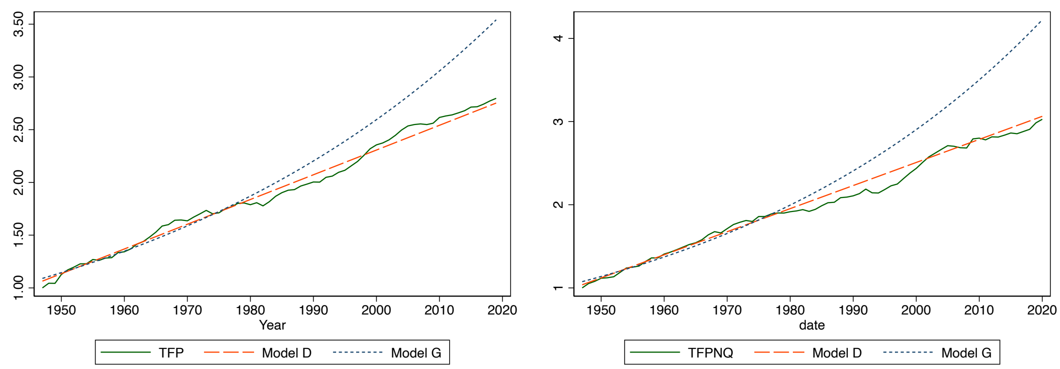
for $t = 1984 : 2019$. Daniela instead fits a linear model and predicts future TFP as $\hat{A}_t^{(D)} = \hat{\alpha}_A + \hat{\Delta}_A t$. Figure 1 shows that Daniela would have made a much better forecast than George. George is puzzled by the TFP slowdown while Daniela does not perceive an obvious long term break in her model (although there are some important medium term deviations that we analyze below). The results obtained from $A_t^{F,NQ}$ and A_t^{BCL} are virtually identical so I focus on only one measure (BCL) for brevity in this section.

Figure 1 reveals a new fact and makes an important empirical point. The new fact is that there is no TFP slowdown in the US according to model D. The important empirical point is that, with realistic values for TFP growth rates, the distinction between models D and G requires at least 10 years of out-of-sample forecasts.

Fact 1. *There is no TFP slowdown in the US according to model D.*

Figure 1: Out-of-Sample TFP Forecasts

(a) BCL Data (b) Fernald Data



Notes: BCL TFP is in based on \$US 2010. Fernald unadjusted TFP, $A_t^{F,NQ} = A_t^F Q_t^{1-\alpha}$. Both are normalized to 1 in 1947. Models are estimated over 1947-1983. The forecast 1984-2019 is out-of-sample. Data source: Fernald (2012) and Bergeaud et al. (2016).

1.3 Capital Accumulation and Labor Productivity

Let us know study the accumulation of capital. Define the capital labor ratio as

$$k_t \equiv K_t / L_t,$$

where, in the BCL data, K_t is the real capital stock and L_t measures hours worked. The first order condition for capital demand in the neoclassical growth model equates the marginal product of capital (MPK) to the user cost (defined as χ). BCL do not consider changes in the user cost and the first order condition is simply

$$k_t^{1-\alpha} = \frac{\alpha}{\chi} A_t. \tag{4}$$

Equation (4) says that the normalized inverse MPK (IMPK) is proportional to A.[3] Model G therefore predicts that $k_t^{1-\alpha}$ grows exponentially, while model D says that it grows linearly. Figure 2 presents the forecasts for $k_t^{1-\alpha}$ based on models D and G with $\alpha = 0.3$, the value used by BCL. For model D we have

$$\mathbb{E}\left[k_t^{1-\alpha}\right] = \hat{\alpha}_{impk} + \hat{\Delta}_{impk} t \tag{5}$$

[3] Users of model G typically interpret equation (4) as saying that capital grows grows exponentially, just like A, as a rate $(1+g)^{1/(1-\alpha)}$. Equivalently, if the model is written with Harrod-neutral technological progress, $Y_t = K_t^\alpha \left(\mathcal{Z}_t H_t\right)^{1-\alpha}$ then capital is proportional to $\mathcal{Z}_t$. I return to these issues in Section 3.

For model G we have the formula in logs. Once again we find that the log-linear model with constant growth widely missed the mark, while the additive model gives a useful forecast. This test is obviously a test of the joint hypothesis of linear TFP growth and a constant user cost (together with Cobb-Douglass capital demand. This last assumption is certainly not correct in many cases, but the data reveals that it may still provide a useful approximation. The main reason for fitting equation (5), however, is to be able to forecast labor productivity (and GDP pre capita).

Once we have a forecast for the capital labor ratio we can use our forecast for TFP to create a forecast for labor productivity λ_t, defined as output per hour:

$$\lambda_t \equiv \frac{Y_t}{L_t} = A_t k_t^{\alpha}. \tag{6}$$

Model D offers a forecast for labor productivity as

$$\hat{\lambda}_t = \left(\hat{\alpha}_A + \hat{\Delta}_A t\right)\left(\hat{\alpha}_{impk} + \hat{\Delta}_{impk} t\right)^{\frac{\alpha}{1-\alpha}}$$

Note that labor productivity is convex in time even under additive growth since it depends on the *product* of both TFP and capital intensity. I could use the forecasts for IMPK and TFP to similarly create a forecast for model G but that would be a straw man since we have already seen that model G fails to predict either A or IMPK. To give model G a chance, I create directly a forecast of labor productivity by fitting the series for $\log(\lambda_t)$ in the first half of the sample. Model G therefore gains two degrees of freedom. Panel (b) in Figure 2 shows that the convex-linear forecast of model D predicts correctly the evolution of labor productivity in the long term. Model G does not.

US growth is better described as additive rather than multiplicative. Instead of stating that the average growth rate of TFP is 1.45%, which is correct but not particularly useful, it is more relevant to say that TFP increases by 0.0245 points each year starting from a normalized value of 1 in 1947.[4] For labor productivity, both the additive growth model and the multiplicative growth model predict an increasing size of productivity increments, but at different speeds. The additive model D predicts that labor productivity increments increase with the square of the time horizon, while the geometric model G predicts exponentially increasing increments. Model G does not describe the data with

[4]The points of TFP need to be expressed in some units. Bergeaud et al. (2016) express TFP at $US 2010 ppp and the annual increase is 0.125 points each year. Alternatively, we can normalized initial TFP to 1 in 1947, as in Fernald (2012). Since TFP in 1947 in Bergeaud et al. (2016) is 5, we obtain $0.125/5 = 0.025$.

Figure 2: Out-of-Sample $IMPK$ and LP Forecasts

(a) IMPK (b) Labor Productivity

Notes: $IMPK = (K/L)^{0.7}$, normalized to 1 in 1947. Models are estimated over 1947-1983. The forecast 1984-2019 is out-of-sample. Labor productivity is real GDP per hour. Data source: Bergeaud et al. (2016).

a constant growth rate. Model D describes the data relatively well with year-on-year increments of about \$1560 per full time worker (\$0.87 per hour, assuming 1800 hours worked in a year) around 2010.

Fact 2. *Postwar US TFP growth is well described by Model D with increments of* $\Delta = 0.0245$ *points each year starting from a normalized value of 1 in 1947. Model D also predicts the correct non-linear evolution of labor productivity.*

1.4 U.S. TFP, 1890-2019

Let us know extend the methodology and the sample, taking into account that trend growth can change over time. Our agents now forecast time varying growth according to a standard exponential smoothing model

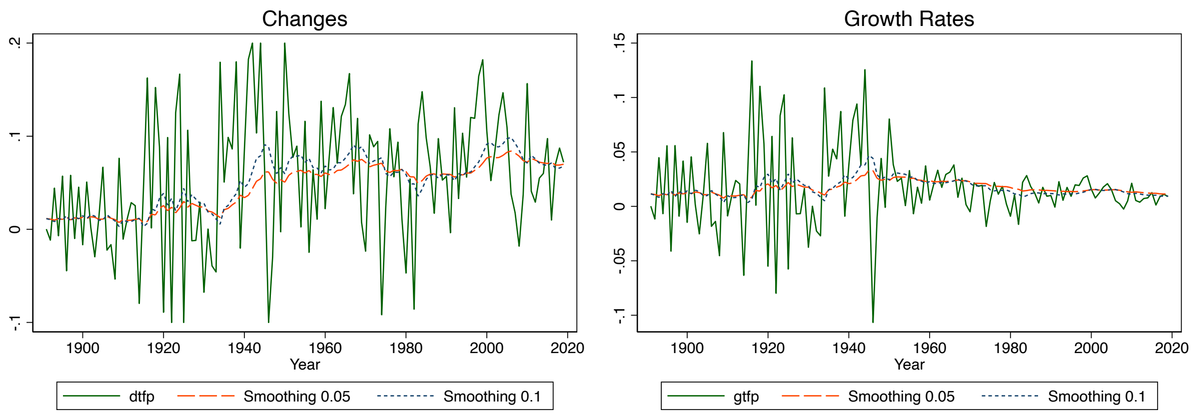
$$\mathbb{E}_t\left[\Delta_{t+1}\right] = (1-\zeta)\,\mathbb{E}_{t-1}\left[\Delta_t\right] + \zeta\Delta_t \tag{7}$$

with $\Delta_t \equiv A_t - A_{t-1}$ for model D and $g_t = A_t/A_{t-1} - 1$ instead of Δ_t for model G.

There are two ways to set the smoothing parameter. One can argue on theoretical grounds that changes in the trend growth rates of TFP are decadal phenomena. This approach suggests values of ζ between 0.05 and 0.1. At 0.05, the sensitivity of the trend estimate to the most recent observation is the same as that of a 20-year moving average. Below 0.05 the model would take too long to adjust to changing trend growth. At 0.1 the

Figure 3: US TFP, 1890-2019

(a) TFP Changes. Model D. (b) TFP Growth Rates. Model G.



Notes: Models are estimated over 1947-1980. The left panel show the prediction of a linear model. The right panel shows the prediction of a log-linear model. US TFP is from the updated work of Bergeaud et al. (2016).

sensitivity would be the same as that of a 10-year moving average. The main advantage of this approach is that it avoids any risk of over-fitting or p-hacking. I will simply report the results for 0.05 and 0.1 (and intermediate values) and see if the results are robust.

Figure 3 shows the raw and smoothed series for $\zeta = 0.05$ and $\zeta = 0.1$. The data is from Bergeaud et al. (2016) and winsorized in the first and last percentiles to remove limit extreme outliers during WW2. The model is initiated over the first 10 observations, 1891 to 1900. As expected the trend growth of the economy changes over this long sample.

The other way to choose ζ is to estimate it in some sample. The advantage is obvious, but the cost is that we waste a sample where we cannot perform out-of-sample tests. Thankfully the two approaches turn out to yield similar results. The smoothing parameter that minimizes the RMSE of one-year forecasts from (7) for the US over 1890-2019 is $\zeta = 0.0664$. If we consider the RMSE of 10-year ahead forecasts, the optimal parameter is $\zeta = 0.055$ (see below for this calculation). In the remaining of the paper I will therefore use $\zeta = 0.05$.

The extreme heteroskedasticity of growth rates is also apparent in Panel (b) of Figure 3. TFP growth rates are much more volatile before than after WW2 – 4.9% vs 1.5% – and volatility declines further after 1980. Romer (1986a) discusses the first fact, McConnell and Perez-Quiros (2000) discuss the second fact, which became know as the great moderation puzzle. These puzzles do not exist in Model D. The standard

Table 1: Volatility of TFP Growth, US 1890-2019

	Model G	Model D
100∗	$\lvert\epsilon_t^g\rvert$	$\lvert\epsilon_t^\Delta\rvert$
(Year-1955)	-0.036	-0.011
t	-6.5	-1.1
Constant	2.49	5.62
t	11.8	15.1
N	129	129
R^2	0.247	0.009

Notes: Dependent variables scaled by 100. For model G the dependent variable is residual growth rate of TFP. For model D the dependent variable is the residual of the first difference of TFP. Data from Bergeaud et al. (2016), US, 1890-2019.

deviation of TFP changes is 0.13 before WW2 and 0.11 since 1947, and the difference is not statistically significant. Formally, define the residuals for model G as

$$\epsilon_t^g = g_t - \mathbb{E}_{t-1}[g_t]$$

and similarly for model D, $\epsilon_t^\Delta = \Delta_t - \mathbb{E}_{t-1}[\Delta_t]$. Table 1 shows that the volatility of TFP growth rates declines significantly over time. I use the absolute value of the unexpected shock to avoid the influence of outliers but the results are similar if I use squared residuals instead, as in ARCH models. Average absolute deviation is 2.5% in the sample, and decline by 3.6 basis point each year on average. Over 50 years the volatility changes by 1.8% which is almost 3/4 of the mean. By contrast the trend is small and insignificant for model D. The change over 50 years is only 10% of the mean.

Fact 3. *There is no volatility puzzle for model D.*

Forecasts Let us know study the forecasting accuracy of the two models as in Figure 1, but instead of performing once test pre/post 1980, I compute real-time rolling estimates and out-of-sample forecasts using (7). Figure 4 shows the 10-year out-of-sample predictions for the level of TFP from model D

$$\mathbb{E}_{t-10}^D[A_t] = A_{t-10} + 10\hat{\Delta}_{t-10} \tag{8}$$

and from model G

$$\mathbb{E}_{t-10}^G[A_t] = A_{t-10}(1+\hat{g}_{t-10})^{10}, \tag{9}$$

Table 2: RMSE for US TFP Forecasts, 1890-2019

Smoothing Parameter	$\zeta = 0.05$		$\zeta = 0.1$	
Forecast Horizon	10 years	20 years	10 years	20 years
Model D	.086	.145	.090	.147
Model G	.107	.209	.114	.237

Notes: US TFP is from the updated work of Bergeaud et al. (2016)

where $\hat{\Delta}_{t-10} \equiv \mathbb{E}_{t-10}[\Delta_{t-9}]$, $\hat{g}_{t-10} = \mathbb{E}_{t-10}[g_{t-9}]$ are the trends estimated 10 years before and I use the fact that $\mathbb{E}_t[\Delta_{t+k}] = \mathbb{E}_t[\Delta_{t+1}]$ for all $k \geq 1$. I then define the long term forecast error as

$$\epsilon_t^{D,G} = \frac{A_t - \mathbb{E}_{t-10}^{D,G}[A_{i,t}]}{\bar{A}},$$

where $\bar{A}$ is the sample average of A. I use this normalization to ease the comparison across datasets where TFP levels are defined in different ways. Table 2 reports the root mean square errors (RMSE) of long term forecasts. Model D outperforms model G in all cases and the relative performance of model D *increases* with the forecast horizon.[5] The main reason is that after a sequence of positive growth rates the multiplicative model extrapolates exponential growth for 10 years, which systematically fails to materialize.

Fact 4. *For US TFP over 1890-2019, model M's long-term forecast errors are 25% to 40% higher than those of Model D.*

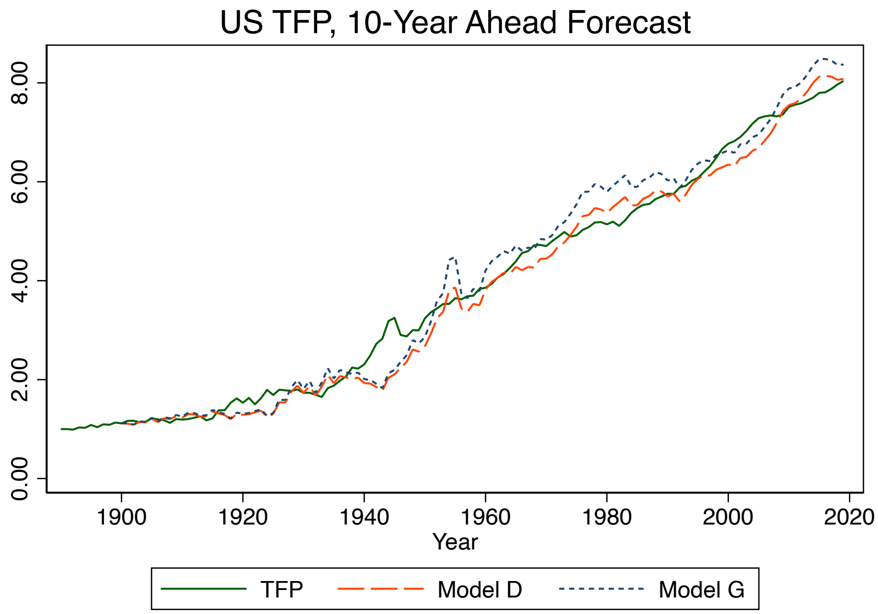
1.5 TFP and GPT

Figure 3 shows that model G is unstable. The estimated trend growth rate is constantly being revised. This is why model G is not useful as a long run growth model. Model D, on the other hand, appears to have only one break. We can formally test this idea following Bai and Perron (2003). The unconstrained test finds one break in the $\Delta[TFP]$ series around 1930 (the point estimate is 1933). We can test H0: no breaks versus H1: break in 1933. The W statistic is 21.72 and the p-value is 0.0. I emphasize, however, that while the existence of a break is clear, the date is really an interval between the late 1920s and WW2.

The date of the break is consistent with Field (2003)'s argument that "*the years 1929–1941 were, in the aggregate, the most technologically progressive of any comparable period in U.S. economic history.*" This period corresponds to the large scale implementation

[5]The same results hold if I compute the RMSE over relative errors $\frac{A_{i,t} - \mathbb{E}_{t-10}^{D,G}[A_{i,t}]}{A_{i,t}}$.

Figure 4: US TFP Forecasts, Long Sample



Notes: Forecast with smoothing parameter 0.05. US TFP is from the updated work of Bergeaud et al. (2016).

of the discoveries of the second industrial revolution: electric light, electric power, and the internal combustion engine, as discussed in Jovanovic and Rousseau (2005). Gordon (2016) points out that it is somewhat surprising that *"much of the progress occurred between 1928 and 1950,"* several decades after the discoveries were made. Following David (1990), he explains the paradox by showing that the 1930s were a period of follow-on inventions, such as the perfection of the piston power-powered aircraft and television, and the increasing quality of machinery made possible by the large increases in available horsepowers and kilowatt-hours of electricity.

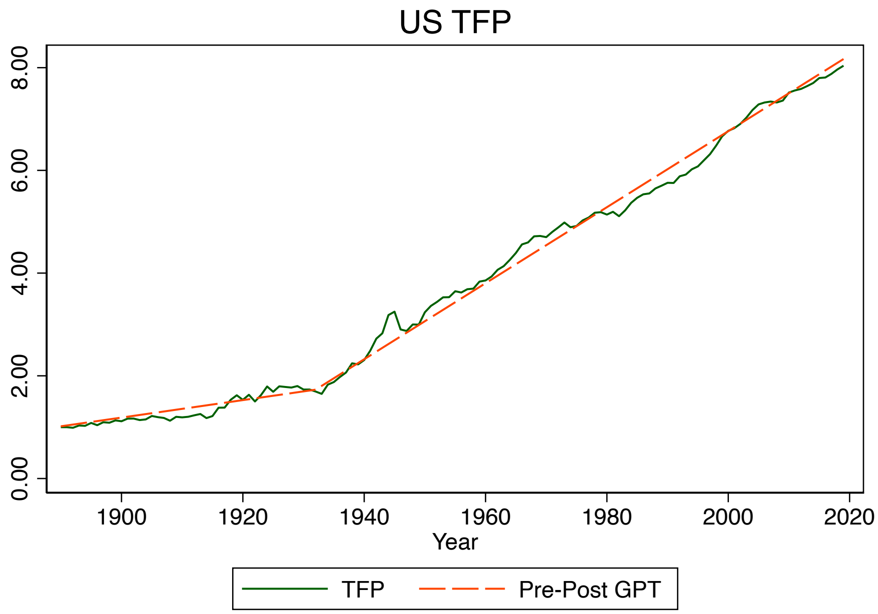
Following these historical insights, figure 5 proposes an interpretation of US TFP from 1890 to 2019, using linear growth with one structural break in 1933 after the electrification revolution.

We can summarize this idea in the following remark, keeping in mind that we normalize US TFP to 1 in 1890.

Fact 5. *From 1890 to 1933, TFP increases by .017 each year until it reaches a level around 1.75 in the early 1930s. From 1933 to 2019 TFP increases by .057 each year (3.3 p.p. of its level in 1933) to reach a level around 8 at the end of the sample.*

Proposition 1 summarizes our results so far.

Figure 5: US TFP under Electrification GPT Interpretation



Notes: US TFP is from the updated work of Bergeaud et al. (2016), normalized to 1 in 1890.

Proposition 1. *Model G does not provide a good description of US TFP growth over 1890-2019, neither for volatility nor for long term forecasts. Model D provides a simple and accurate description as*

$$A_t - A_{t-1} = \bar{\Delta}_{GPT} + \epsilon_t$$

where $A_{1890} = 1$, ϵ_t *is iid with a mean absolute deviation of 0.056,* $\bar{\Delta}_{GPT} = 0.017$ *between 1890 and 1933, and* $\bar{\Delta}_{GPT} = 0.057$ *from 1933 to 2019.*

I will discuss the pre-1890 period in Section 4 but it is useful at this point to emphasize that backcasting is the not the same as forecasting. My results show that the D-model offers better forecasts that the G-model, at least over a few decades. But the GPT model does not predict linear backcasts, because conditional on high productivity today, we know there must have been a break in the not-too-distant past. Thus the model does not predict that TFP was zero in 1831 (1/0.017=59 years from 1890). Instead it says that there must have been a break sometime in the 19th century. Section 4 shows that the data is consistent with this prediction.

Table 3: RMSE for 23 Countries, BCL Sample

Sample	1890-2019		1950-2019	
Parameter	$\zeta = 0.05$	$\zeta = 0.1$	$\zeta = 0.05$	$\zeta = 0.1$
Model D	.130	.128	.102	.103
Model G	.171	.168	.162	.145
N. Obs.	23	23	23	23

Notes: Data from Bergeaud et al. (2016). Sample 1950-2019.

2 Country-Level International Evidence

Bergeaud et al. (2016) provide data for 23 countries.[6] The trend growths are estimated with the recursive learning model (7) with parameter $\zeta = 0.05$ and $\zeta = 0.1$. As before, all the forecasts are out-of-sample. For each country $i = 1 : 23$ and each year t I compute the forecast errors as

$$\epsilon_{i,t}^{D,G} = \frac{A_{i,t} - \mathbb{E}_{t-10}^{D,G}\left[A_{i,t}\right]}{\bar{A}_i},$$

where $\bar{A}_i$ is the country sample average and the expectation are taken under models D and G. Finally, I compute the root mean square error for each country as

$$\mathtt{RMSE}_i^{D,G} = \sqrt{\frac{1}{T}\sum_{t=1}^{T}\left(\epsilon_{i,t}^{D,G}\right)^2}.$$

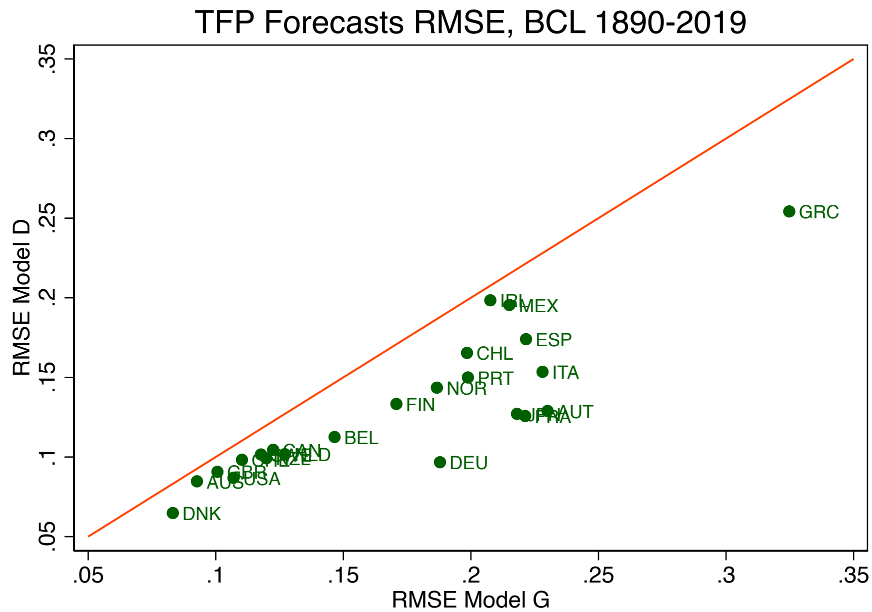
2.1 Long-Sample, 1890-2019

I first run the model over the whole sample, 1890-2019, initializing over the first 10 years. Figure 6 shows that the D-model out-performs the G-model for every single country in the BCL sample.

Table 3 summarizes the average performance of models D and G. The differences are larger than in Table 2 because many countries experience more volatile growth sequences than the US, which makes it easier to separate the two models. Model D over-performs model G by 30% to 60%.

[6] Australia, Austria, Belgium, Canada, Switzerland, Chile, Germany, Denmark, Spain, Finland, France, United Kingdom, Greece, Ireland, Italy, Japan, Mexico, Netherlands, Norway, New Zealand, Portugal, Sweden and United States. The sample covers 1890–2019. The main variables are GDP, labor, and capital. Labor is constructed from data on total employment and working time. Capital is constructed by the perpetual inventory method applied equipment and buildings.

Figure 6: TFP Forecast Errors, BCL 1890-2019



Notes: US TFP is from the updated work of Bergeaud et al. (2016).

2.2 Post-War Sample

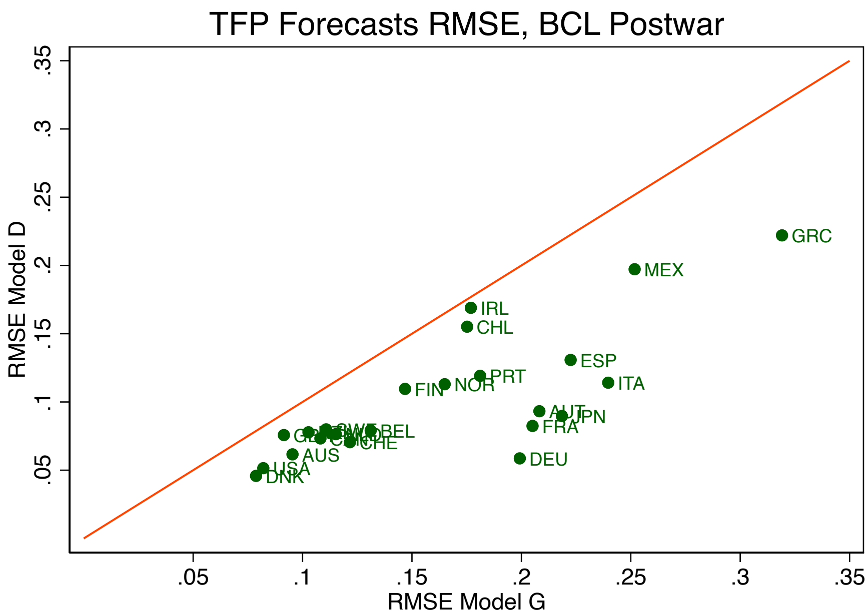
I run the model separately for the post war period because several countries (e.g. Japan, Germany) experience large shocks during the 1940s which may render the forecasts from the exponential model unstable. Figure 7(a) shows the RMSE of TFP forecasts in the two datasets. Model D performs better than model G in all cases.

I also use the OECD MFP database as a robustness check in Figure 7(b). The data covers 24 countries and starts in 1985 for most, and later for some. Because the time series are much shorter it is more difficult to tell the models apart and some countries are bunched close to the 45 degree line. Nonetheless, model G never performs better than model D, and often performs worse. Perhaps the most interesting case is that of Korea, which is not in the BCL sample and has experienced strong growth over the past 30 years. It turns out that Korean TFP growth is very linear.

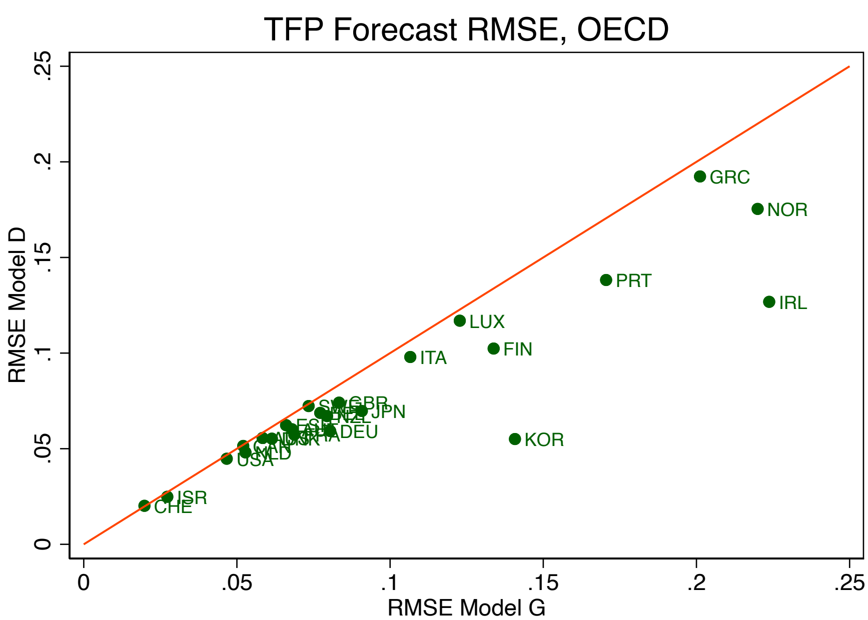
Remark 1. In a sample of developed countries, TFP growth is better described by model D than by model G.

Figure 7: TFP Forecast Errors, Post War

(a) BCL 1950-2019



(b) OECD, post-1985



Notes: Model G on the horizontal axis, model D on the vertical axis. Out-of-sample, 10-year forecasts with smoothing parameter 0.05. Data from Bergeaud et al. (2016). Sample 1950-2019.

3 Implications for Long Term Growth

The goal of this section is to highlight the most important features of additive growth in the aggregate. I leave industry and firm dynamics for future research. To draw long-run implications from the theory we must first revisit the exact nature of the production function.

3.1 Finding Linearity

As explained in Barro and Sala-i-Martin (2004), balanced growth requires labour-augmenting (Harrod-neutral) technology

$$Y_t = F\left(K_t, \mathcal{Z}_t L_t\right) \tag{10}$$

where $\mathcal{Z}_t$ is labor-augmenting, or Harrod-neutral, technological progress. We need an exact mapping between this functional form and the evidence discussed so far. This is important even if F is Cobb-Douglas, as Fernald (2012) and Bergeaud et al. (2016) assume. In that case we can of course renormalize our productivity measure as $A = \mathcal{Z}^{1-\alpha}$ so that (10) becomes $Y_t = A_t K_t^{\alpha} L_t^{1-\alpha}$, but this does not answer the question of whether A or $\mathcal{Z}$ (or perhaps neither) is best described as a linear process. To be concrete, if $\mathcal{Z}$ is linear then A is concave in time. If instead A is linear then $\mathcal{Z}$ is convex in time. These distinctions matter for long-run growth.

Fernald (2012) provides a good framework to discuss these issues. He writes the following production function

$$Y_t = A_t^F K_t^{\alpha} \left(Q_t H_t\right)^{1-\alpha},$$

where A_t^F Fernald's headline TFP measure, capital services K adjusted for variable utilization are constructed from disaggregated series on structures, equipments and IPs, and Q_t is a labor-quality index constructed from rolling Mincer wage regressions in the Current Population Survey. I consider 5 hypotheses:

- (i) GA: the Hicks-neutral series net of labor quality improvements, A_t^F, features constant *exponential* growth.
- (ii) DA: A_t^F features constant *additive* growth.
- (iii) DAE: $A_t^{F,NQ} = A_t^F Q_t^{1-\alpha}$, the Hicks-neutral TFP *including* educational im-

provements is additive. Whether or not one should net out the effect of education when measuring TFP depends on the question at hand. Solow (1957) explains that he uses "the phrase "technical change" as a short-hand expression for any kind of shift in the production function. Thus [..] improvements in the education of the labor force, and all sorts of things will appear as "technical change"." If one follows this line of reasoning, then $A_t^{F,NQ}$ is the more relevant concept.

- (iv) DZ: $\left(A_t^F\right)^{\frac{1}{1-\alpha}}$, the Harrod-neutral series adjusted for labor quality, is additive. If the true model is $Y_t = F\left(K_t, \mathcal{Z}_t Q_t H_t\right)$ with $\mathcal{Z}_t$ additive, then we should find that $\left(A_t^F\right)^{\frac{1}{1-\alpha}}$ is additive.

- (v) DZE :$\left(A_t^F\right)^{\frac{1}{1-\alpha}} Q_t$, the Harrod-neutral series including educational improvements is additive. If the true model is $Y_t = F\left(K_t, \mathcal{Z}_t H_t\right)$ with $\mathcal{Z}_t$ additive, then we should find that $\left(A_t^F\right)^{\frac{1}{1-\alpha}} Q_t$ is additive.

Table 4 shows the estimates for the US from Fernald's data, together with one estimate from the BCL data for comparison. Column (i) documents the well-know TFP "slow-down" which is a puzzle for the exponential growth model (the puzzle is the same for all G models irrespective of the labor quality or Harrod-neutral adjustments, omitted for brevity). The sample average TFP growth is 1.3% per year, but loses 2.6 basis point each year, from around 2% in the early 1950s down to only 0.5% after 2010. Column (ii) shows that the puzzle is much reduced, but not entirely eliminated, by the AF specification.

Column (iii) and (iv) show that DAE and DZ are two equally plausible way to characterize additive growth. DZ assume linear labor-augmenting productivity applied to quality adjusted labor $Q_t H_t$. DAE folds educational improvements into Hicks-TFP growth instead of netting them out. Column (v) shows that doing both adjustments simultaneously might be excessive in the US. Column (vi) shows that the BCL series is similar to the Hicks-neutral series based on raw labor in (iii), which is consistent with our discussion in Section 1.

The data therefore suggests that models (iii) and (iv) provide reasonable descriptions of additive productivity growth. An important point is that both predict increasing improvements in labor productivity, but for slightly different reasons. In model DZ, $\mathcal{Z}_t$ is linear, which by itself generates linear labor productivity, but it applies to an increasingly qualified quantity of labor. In model DAE the combination of educational and non educational improvements generates Hicks-neutral additive growth.

Table 4: Trends in US TFP Growth

	(i)	(ii)	(iii)	(iv)	(v)	(vi)
Specification	GA	DA	DAE	DZ	DZE	BCL
$100 * \Delta\,[.]$	$\log A_t$	A_t	$A_t Q_t^{1-\alpha}$	$A_t^{\frac{1}{1-\alpha}}$	$A_t^{\frac{1}{1-\alpha}} Q_t$	A_t^{BCL}
(Year-1983)	-0.026	-0.015	-0.002	-0.005	0.038	-0.003
t	-3.3	-1.1	-0.1	-0.2	1.1	-0.3
Constant	1.311	2.107	2.758	4.131	5.740	2.498
t	7.9	7.2	8.5	6.7	7.9	9.9
N	72	72	72	72	72	72
R^2	0.133	0.017	0.000	0.000	0.017	0.001

Notes: US, 1947-2019. A_t refers to Fernald's measures of TFP, adjusted for labor quality Q, and normalized to 1 in 1947. In column (iii) the labor quality adjustment is added back to the TFP measure. All dependent variables scaled by 100. In (i) the dependent variable is the growth rate of TFP. In (ii-vi) the dependent variable is the first difference of TFP. Data from Fernald (2012) in (i-v) and Bergeaud et al. (2016) in (vi). The independent variable is year minus sample mean (1983) so that the constant can be readily interpreted as the sample average growth in percent for (i) and in percentage points of the 1947 value in the other columns.

There is no clear statistical reason to prefer the Hicks-additive model (iii) to model (iv), but model (iii) has two practical advantages. It is more directly comparable to the BCL TFP series, and it requires the forecast of only one factor (A_t) instead of two ($\mathcal{Z}_t$ and Q_t^L).

Fact 6. *A first order account of growth in the US private sector since 1947 is that output is a Cobb-Douglass function of capital and hours worked $Y_t = A_t K_t^{\alpha} H_t^{1-\alpha}$, and Hicks-TFP A_t – including educational improvements and normalized to 1 in 1947 – increases by about 2.76 percentage points each year.*

After twenty years, TFP is 1.55, after forty years it is 2.1, and so on. This model accounts well for the evolution of TFP in the US since 1947. There is no slowdown in TFP increments. The point estimates of -0.002 is rather precisely estimated at 0.

3.2 Theoretical Properties

Let us now turn to the theoretical dynamics of the Hicks-additive model. I use continuous time to simplify the notations. Output is $Y_t = A_t K_t^{\alpha} H_t^{1-\alpha}$, TFP grows according to $\frac{dA_t}{dt} = \Delta$, and capital accumulates as

$$\frac{dK_t}{dt} = I_t - \delta K_t.$$

Hours grow at the constant population growth rate g_n: $\frac{dH_t}{dt} = g_n H_t$.

Solow-Swan Dynamics Let us start with a textbook model with a fixed saving rate s: $I_t = sY_t$. Define $\mathcal{Z}_t = A_t^{\frac{1}{1-\alpha}}$ and the scaled capital stock as

$$\kappa_t \equiv \frac{K_t}{\mathcal{Z}_t H_t}$$

to obtain the usual differential equation

$$\dot{\kappa}_t = s\kappa_t^\alpha - \left(\delta + g_n + \frac{\dot{\mathcal{Z}}_t}{\mathcal{Z}_t}\right)\kappa_t \tag{11}$$

with $\frac{\dot{\mathcal{Z}}_t}{\mathcal{Z}_t} = \frac{1}{1-\alpha}\frac{\dot{A}_t}{A_t}$. The exponential growth model predicts that $\frac{\dot{A}_t}{A_t}$ is constant. The additive growth model predicts that $\frac{\dot{A}_t}{A_t} = \frac{\Delta}{A_t}$ declines over time for a given $\Delta > 0$. Since $\lim_{t\to\infty} A_t = \infty$ we have the following proposition.

Proposition 2. *The long-run balanced growth path is characterized by* $\kappa_\infty = \left(\frac{s}{\delta+g_n}\right)^{\frac{1}{1-\alpha}}$*. The capital labor ratio grows as* $k_t = \kappa_\infty A_t^{\frac{1}{1-\alpha}}$ *and labor productivity (or GDP per capita) as* $\lambda_t = \kappa_\infty^\alpha A_t^{\frac{1}{1-\alpha}}$*. The increments of* k_t *and* λ_t *increase to infinity,* $\lim_{t\to\infty}\frac{d\lambda_t}{dt} = \infty$*, but their growth rates converge to zero* $\lim_{t\to\infty}\frac{d\log\lambda_t}{dt} = 0$.

Let me now discuss a few implications and extensions of the model.

Long Term Growth Note that for large t we have $\frac{d\lambda_t}{dt} \approx \frac{1}{1-\alpha}\kappa_\infty^\alpha (A_0 + \Delta t)^{\frac{\alpha}{1-\alpha}}\Delta$. This shows that labor productivity, and thus living standards, grow as an increasing pace when we assume Hicks-linear growth. If we instead assume (counter-factually as discussed above) Harrod-linear growth, $\dot{\mathcal{Z}}_t = \Delta$, then improvements in living standards would not go to infinity but instead converge to a finite limit: $\frac{d\lambda_t}{dt} \to \kappa_\infty^\alpha\Delta$. Models DAE (iii) and DZE (v) therefore make different predictions about long run growth. The good news is that the available evidence seems to support model DAE. One caveat, however, is that model DAE is linear because we include educational achievements into TFP growth. If the educational achievements of the 20th century as documented by Goldin and Katz (2008) cannot be repeated, growth could fall to that implied by the DA model. Similarly, Hsieh et al. (2019) show that human capital misallocations have decreased over the past 60 years. They argue that "*a substantial pool of innately talented women and black men in 1960 were not pursuing their comparative advantage*," and that the improved allocation of talent can explain 20 to 40 percent of labor productivity growth. If this improvement cannot be repeated our estimate of long run Δ could be biased upward.

Convergence Transitional dynamics are essentially the same as in the standard model, so results on conditional convergence, discussed for instance in Barro and Sala-i-Martin (2004) are unchanged. Unconditional convergence depends on how Δ's vary across countries and over time. Permanent differences in Δ predict infinitely increasing inequality even though growth rates converge to zero in all countries. Changes in Δ over time predict time-varying catch. For instance, if a country manages to permanently increase its Δ, its catch up would be initially quick but then slower.

Neoclassical Production Function The results in Proposition 2 generalize beyond the Cobb-Douglas case. Define $\mathcal{Z}_t$ as the Harrod growth in a neoclassical production function $Y_t = F\left(K_t, \mathcal{Z}_t H_t\right)$. The dynamic equation becomes

$$\dot{\kappa}_t = sf\left(\kappa_t\right) - \left(\delta + g_n + \frac{\dot{\mathcal{Z}}_t}{\mathcal{Z}_t}\right)\kappa_t$$

where $f\left(\kappa\right) \equiv F\left(\kappa, 1\right)$. As before the limit solves $sf\left(\kappa_\infty\right) = \left(\delta + g_n\right)\kappa_\infty$. Define

$$\alpha_\infty \equiv \lim_{\kappa \to \kappa_\infty} \alpha\left(\kappa\right)$$

where $\alpha\left(\kappa\right) \equiv \frac{KF_K}{F}$ estimated at the point $\kappa = \frac{K}{\mathcal{Z}H}$. Thus α_∞ is simply the capital elasticity (or capital share) estimated at κ_∞. This model is consistent with the evidence on additive Hicks productivity growth if and only if $\mathcal{Z}\left(A\right) \approx A^{\frac{1}{1-\alpha_\infty}}$ where A is additive.

Ramsey Model and Interest Rates Let me finally discuss the case where savings are endogenous. I again assume a textbook model where infinitely-lived households have CRRA preferences and fixed labor supply. The equilibrium is pinned down by capital accumulation and the households' Euler equation (and the usual transversality condition, omitted here):

$$\dot{\kappa}_t = f\left(\kappa_t\right) - \hat{c}_t - \left(\delta + g_n + \frac{\dot{\mathcal{Z}}_t}{\mathcal{Z}_t}\right)\kappa_t,$$

$$\frac{\dot{\hat{c}}_t}{\hat{c}_t} = \sigma\left(f'\left(\kappa_t\right) - \delta - \rho\right) - \frac{\dot{\mathcal{Z}}_t}{\mathcal{Z}_t},$$

where $\hat{c}_t = \frac{C_t}{H_t \mathcal{Z}_t}$ is normalized consumption per capita, σ is the EIS and ρ the rate of time preference. As before, we have $\lim_{t\to\infty} \frac{\dot{\mathcal{Z}}_t}{\mathcal{Z}_t} = 0$ so the long-term balanced growth path is given by

$$f'\left(\kappa_\infty\right) = \delta + \rho$$

and

$$\hat{c}_\infty = f\left(\kappa_\infty\right) - \left(\delta + g_n\right)\kappa_\infty$$

All per capital variables grow with $\mathcal{Z}_t$. For instance, long run per capita consumption is $c_t = \hat{c}_\infty \mathcal{Z}_t$. What is interesting, however, is the behavior of interest rates. The model features decreasing growth rates, so if we assume CRRA preferences, the model predicts that interest rates fall over time and eventually converge to ρ.

3.3 Endogenous Growth

The additive model can be cast as a semi-endogenous growth model. I follow Jones (2021a) and ignore capital accumulation as it is not crucial here. I assume first that population is constant at N. People are employed in production L or in research R and the labor resource constraint is $R+L = N$, and as in Jones (2021a) I assume $R = \kappa N$ for some constant κ. Output is given by $Y = AL$ so output per capita is $y = \frac{Y}{N} = (1-\kappa)A$. The simplest semi-endogenous growth equation is then

$$\frac{dA}{dt} = \Gamma\left(R\right) = \Gamma\left(\kappa N\right). \tag{12}$$

This model delivers additive TFP growth given κ and N. If we replaced $Y = AL$ with a neoclassical production function we would find that labor productivity is convex as before. For a given function Γ we can endogenize κ by equating private returns to innovation with the labor market wage, as in standard endogenous growth models.

There are two important issues. The first issue is that Γ changes over time following the discovery of a GPT. Equation (12) holds within a GPT period but not across GPTs. A corollary of that issue is that we need to take a stand on the persistence of a GPT shock. Should we assume that a GPT permanently increases the (potential) growth of the economy? Or should we assume that the impact on Δ depreciates over time? One could speculate that the slowdown of the late 1970s in Figure 5 reflect the waning impact of the initial electricity revolution and the pickup in the late 1980s the impact of IT. Section 4 makes some progress on these questions but this is an important issue for future research.

The second issue is that population growth can overturn the additive growth prediction of equation (12). If $R_t = \kappa N_t$ grows over time then equation (12) says that $\frac{dA}{dt}$ will not be constant. This problem is the same in most growth models (Jones, 2021a) and not specific to the additive model. One can restore additive growth by assuming strongly decreasing returns to idea production.[7] Define knowledge as B with $\frac{dB}{dt} = \gamma R$. If $A = \log(B)$, then $dA = \frac{dB}{B} = g_n$ is constant. Why would this be the case? Jones (2021b) provides a possible micro-foundation based on combinatorial growth. Suppose ideas are drawn randomly and only the best idea matters. If the number of draws grows exponentially (e.g. because of growth in the number of researchers) and if we draw from an exponential distribution, then Jones (2021b) shows that the maximum draw grows linearly over time.

4 Growth Before 1890

The third issue is the persistence of a GPT shock. If we assume that a GPT permanently increases the (potential) growth of the economy then

$$\tilde{\Delta}_T = \sum_{\tau_i < T} \delta_{\tau_i}$$

where τ_i is the period where GPT i is discovered. The key point of this specification is that, within a GPT era, it predicts constant increments and a slowdown in growth rates.

5 Conclusion

Additive growth is important for the valuation of long term assets, such as stocks or pensions. The model predicts falling growth rates and falling interest rates so valuation effect will depend on preferences. Another is convergence of income across countries, along the lines of Barro (1991), as briefly discussed above.

For models of endogenous growth additive growth speaks to the appropriate functional form of the innovation technology. Bloom et al. (2017) recast the TFP slowdown puzzle in terms of decreasing returns to R&D. If growth is additive it seems natural to think that ideas are also additive, which would solve the slowdown puzzle, but not

[7] The insights in this paragraph are from Chad Jones. The potential mistakes are mine.

necessarily the research effort puzzle. A more interesting question, however, may be to think about circumstances when knowledge creation is likely to be additive and when it is likely to be geometric.

Additive growth has implications for industry dynamics and structural transformation as in Baumol (1967), and for firms dynamics as in Luttmer (2007) and Gabaix (2011). Philippon (2022b) and Philippon (2022a) provide some early evidence on these issues.

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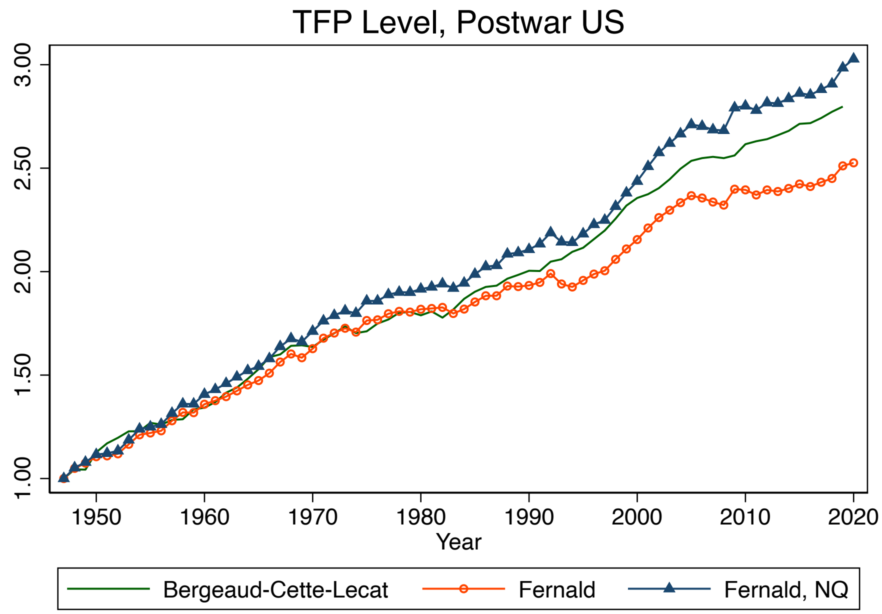
Figure 8: US TFP Levels



Notes: TFP levels, A^{bcl}, A_t^f, and A_t^n. Data from Fernald (2012) and Bergeaud et al. (2016).

Appendix

A Three Measures of Post-War US TFP